

Extrinsic curvature and the Einstein constraints

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The Einstein initial-value equations in the extrinsic curvature (Hamiltonian) representation and conformal thin sandwich (Lagrangian) representation are brought into complete conformity by the use of a decomposition of symmetric tensors which involves a weight function. In stationary spacetimes, there is a natural choice of the weight function such that the transverse traceless part of the extrinsic curvature (or canonical momentum) vanishes.

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I. INTRODUCTION

In this paper we introduce a new decomposition of symmetric tensors and apply it to the construction of extrinsic curvature in the initial-value equations of general relativity. Our results improve previous work that dealt with extrinsic curvature [1,2]. The new findings are consistent with the conformal thin sandwich equations, which involve no tensor decompositions [3].

The thin sandwich and extrinsic curvature formulations differ in whether the velocity or the momentum associated with the conformal spatial metric is specified. The corresponding Lagrangian and Hamiltonian pictures of dynamics must certainly agree, and we find a clear and explicit form of such agreement in our analysis. This would not occur without the presence of a weight function, which must be the lapse function, in the new decomposition.

The presence of the lapse function is also crucial for another result of this decomposition: For stationary spacetimes, there is a natural way to choose the lapse function which results in a *vanishing* transverse traceless part of the extrinsic curvature.

Finally, we extend the conformal thin sandwich equations by giving the velocity $\dot{\tau}$ of the mean curvature $\tau = \text{Tr } \mathbf{K}$ in order to determine the lapse function.

II. NEW TENSOR SPLITTINGS

We will define and apply a new class of covariant decompositions of the extrinsic curvature¹ of a three-dimensional hypersurface \mathcal{M} . It should be noted that they apply to *any* symmetric $\binom{2}{0}$ or $\binom{0}{2}$ tensor for *any* dimension $m \geq 3$. Generalization to $m > 3$ is straightforward (the case $m = 2$ is special). The decompositions use the geometry of \mathcal{M} , that is, the metric \bar{g}_{ij} and the derivative $\bar{\nabla}$, together with a positive scalar function $\bar{\sigma}$ to be specified later.

Given (\mathcal{M}, \bar{g}) , we first remove the trace of the extrinsic curvature \bar{K}^{ij} :

$$\bar{K}^{ij} = \bar{A}^{ij} + \frac{1}{3} \tau \bar{g}^{ij} \quad (2.1)$$

where \bar{A}^{ij} is traceless and $\tau = \bar{\tau} = \bar{g}_{ij} \bar{K}^{ij}$ is the trace of \bar{K}^{ij} , called the “mean curvature.” Note that here, and in the sequel, overbars are used to distinguish spatial tensors from their conformally transformed counterparts. For example, $\bar{g}_{ij} = \varphi^4 g_{ij}$, $\varphi > 0$, where \bar{g} and g are both spatial metrics. Quantities with overbars have physical values.

The next step is to decompose the traceless symmetric tensor \bar{A}^{ij} covariantly. Like previous studies, for example [4–6], we attempt to find a covariantly defined divergence-free part of \bar{A}^{ij} with zero trace. We want to stress that this so-called “transverse-traceless” (TT) part of \bar{K}^{ij} (or \bar{A}^{ij}) is *not unique*. There are infinitely many mathematically well-defined ways to extract such a piece of the original tensor, for example by varying the choice of a weight function [see Eq. (2.2) below]. However, it is possible that by imposing *geometrical or physical requirements*, we can make the TT part and the other parts unique.

We write \bar{A}^{ij} as a sum of a TT part and a weighted longitudinal (vector) traceless part as follows:

$$\bar{A}^{ij} = \bar{A}_{TT}^{ij} + \bar{\sigma}^{-1} (\bar{\mathbb{L}}Y)^{ij} \quad (2.2)$$

where the inverse weight function $\bar{\sigma}$ is a uniformly positive and bounded scalar on \mathcal{M} : $0 < \epsilon \leq \bar{\sigma} < \infty$, $\epsilon = \text{const}$. We define [4–6]

$$(\bar{\mathbb{L}}Y)^{ij} = \bar{\nabla}^i Y^j + \bar{\nabla}^j Y^i - \frac{2}{3} \bar{g}^{ij} \bar{\nabla}_k Y^k, \quad (2.3)$$

which is proportional to the Lie derivative with respect to Y^i of a unimodular inverse metric conformally related to \bar{g}^{ij} ; thus, take the Lie derivative of $[\det(\bar{g}_{kl})]^{1/3} \bar{g}^{ij}$. Expression (2.3) is zero for $Y^i \neq 0$ if and only if Y^i is a conformal Killing vector of the metric, if such a symmetry exists. This also suggests that if we make a conformal transformation $\bar{g}_{ij} \rightarrow \varphi^{-4} \bar{g}_{ij} = g_{ij}$, $\varphi > 0$, then Y^i will not be conformally scaled: $Y^i \rightarrow Y^i$. We will adopt this scaling rule below; because of its simplicity, we omit overbars on vectors like Y^i .

The decomposition is effected by solving for Y^i in

¹We give a general definition of extrinsic curvature in the appendix, which also details our conventions. See in particular Eqs. (A10) and (A18).

$$\bar{\nabla}_j[\bar{\sigma}^{-1}(\bar{\mathbb{L}}Y)^{ij}] = \bar{\nabla}_j\bar{A}^{ij} \quad (2.4)$$

and then setting

$$\bar{A}_{TT}^{ij} \equiv \bar{A}^{ij} - \bar{\sigma}^{-1}(\bar{\mathbb{L}}Y_{(\text{solution})})^{ij}, \quad (2.5)$$

where the given $\bar{\sigma}$ and the solution for Y^i are inserted.

The operator on the left side of Eq. (2.4), $(\bar{\Delta}_{L,\bar{\sigma}}Y)^i$, is similar to the vector Laplacian $(\bar{\Delta}_L X)^i \equiv \bar{\nabla}_j(\bar{\mathbb{L}}Y)^{ij}$, which is solvable on compact manifolds and on asymptotically flat manifolds, given certain asymptotic conditions. The conditions needed are not onerous. Inserting the weight factor leaves the operator in “divergence form,” and does not affect the formal self-adjointness of the weighted vector Laplacian. The calculation of self-adjointness is carried out in the natural measure $\mu_g = \sqrt{g} d^3x$. On the other hand, the two pieces of \bar{A}^{ij} in Eq. (2.2) are formally orthogonal in the positive measure $\bar{\mu}_{\bar{g},\bar{\sigma}} = \bar{\sigma}\sqrt{\bar{g}} d^3x = \bar{\sigma}\mu_{\bar{g}}$.

Note that Eq. (2.4) must be supplemented with *boundary conditions* in the case where \mathcal{M} is not compact without boundary and that the solution of Eq. (2.4) and \bar{A}_{TT}^{ij} will depend on these boundary conditions [7]. This caveat about boundary conditions is very serious in practice, where there may be excised regions of \mathcal{M} , and where \mathcal{M} may have no asymptotic region, though it models a space with Euclidean asymptotic conditions. There is no uniqueness without boundary conditions.

The conformal properties of the new splittings assigned here are very important for application to the Einstein constraints, but they are also very interesting in themselves.

We take

$$\bar{A}_{TT}^{ij} = \varphi^{-10} A_{TT}^{ij}, \quad (2.6)$$

$$\bar{Y}^i = Y^i, \quad (2.7)$$

$$\bar{\sigma} = \varphi^6 \sigma. \quad (2.8)$$

Transformations (2.6) and (2.7) should be familiar. We adopted Eq. (2.8) to obtain correct divergence relations in the sequel. It also maintains the correct transformations when we set $\bar{\sigma} = 2\bar{N}$ and $\sigma = 2N$ in Sec. IV below.

Next we use $\bar{g}_{ij} = \varphi^4 g_{ij}$ with its concomitant transformation rule

$$\bar{\Gamma}_{jk}^i = \Gamma_{jk}^i + 2\varphi^{-1}(\delta_j^i \partial_k \varphi + \delta_k^i \partial_j \varphi - g_{jk} g^{il} \partial_l \varphi) \quad (2.9)$$

for the Christoffel symbols of \bar{g}_{ij} and g_{ij} . If we were to consider a general symmetric tensor \bar{T}^{ij} in three dimensions, with $\bar{g}_{ij} = \varphi^4 g_{ij}$, then $\bar{T}^{ij} = \varphi^x T^{ij}$ yields

$$\begin{aligned} \bar{\nabla}_j \bar{T}^{ij} &= \varphi^x [(\nabla_j T^{ij}) + (x+10)(\partial_j \log \varphi) T^{ij} - 2(\partial_j \log \varphi) \\ &\quad \times (g^{ij} g_{kl} T^{kl})]. \end{aligned} \quad (2.10)$$

This shows why we choose $x = -10$ in Eq. (2.6), as well as the zero trace, $g_{ij} \bar{A}_{TT}^{ij} = 0$.

From Eqs. (2.6)–(2.10), we find, besides $\bar{\nabla}_j \bar{A}_{TT}^{ij} = \varphi^{-10} \nabla_j A_{TT}^{ij} = 0$,

$$(\bar{\mathbb{L}}Y)^{ij} = \varphi^{-4} (\mathbb{L}Y)^{ij}, \quad (2.11)$$

$$\bar{\sigma}^{-1}(\bar{\mathbb{L}}Y)^{ij} = \varphi^{-10} [\sigma^{-1}(\mathbb{L}Y)^{ij}]. \quad (2.12)$$

Therefore,

$$\bar{A}^{ij} = \bar{A}_{TT}^{ij} + \bar{\sigma}^{-1}(\bar{\mathbb{L}}Y)^{ij} = \varphi^{-10} [A_{TT}^{ij} + \sigma^{-1}(\mathbb{L}Y)^{ij}] = \varphi^{-10} A^{ij}, \quad (2.13)$$

so that the decomposition (2.2) is preserved under conformal transformations. The scaling $\bar{A}^{ij} = \varphi^{-10} A^{ij}$ is not postulated, but follows from Eqs. (2.6)–(2.8). Furthermore,

$$\bar{K}^{ij} = \varphi^{-10} [A_{TT}^{ij} + \sigma^{-1}(\mathbb{L}Y)^{ij}] + \frac{1}{3} \varphi^{-4} g^{ij} \tau. \quad (2.14)$$

We note that the conformal quantities A_{TT}^{ij} and $\sigma^{-1}(\mathbb{L}Y)^{ij}$ are orthogonal in the rescaled measure $\sigma\sqrt{g} d^3x = \sigma\mu_g = \mu_{g,\sigma}$.

III. EINSTEIN CONSTRAINTS

The extrinsic curvature \bar{K}^{ij} is uniquely related to the canonical momentum

$$\pi^{ij} = (\text{const}) \bar{g}^{1/2} (\tau \bar{g}^{ij} - \bar{K}^{ij}). \quad (3.1)$$

Therefore the present study is devoted to a construction of quantities belonging to the canonical or Hamiltonian picture of dynamics, while the conformal thin sandwich equations [3] belong to the velocity phase space or Lagrangian picture. It is essential to understand the constraints in *both* pictures, and the pictures must be geometrically and physically consistent.

Elaboration of the constraints in vacuum

$$\bar{\nabla}_j(\tau \bar{g}^{ij} - \bar{K}^{ij}) = 0, \quad (3.2)$$

$$\bar{K}_{ij} \bar{K}^{ij} - \tau^2 - \bar{R} = 0, \quad (3.3)$$

where \bar{R} is the “scalar curvature” or “trace of the Ricci tensor” of $(\mathcal{M}, \bar{g}, \bar{\nabla})$, is facilitated by displaying them as

$$\bar{\nabla}_j \bar{A}^{ij} = \frac{2}{3} \bar{g}^{ij} \partial_j \tau, \quad (3.4)$$

$$\bar{A}_{ij} \bar{A}^{ij} - \frac{2}{3} \tau^2 - \bar{R} = 0. \quad (3.5)$$

The form (3.4) and (3.5) of the constraints is the better one for beginning formulating the constraints in the thin sandwich decomposition or in the extrinsic curvature form.

It has been pointed out [1,8] that the standard transverse traceless tensor decomposition [5,6],

$$\bar{T}^{ij} - \frac{1}{3} \bar{g}^{ij} (\bar{g}_{kl} \bar{T}^{kl}) = \bar{T}_{TT}^{ij} + (\bar{\mathbb{L}}V)^{ij} \quad (3.6)$$

has the property that extracting the TT part of a symmetric tensor does not commute with conformal transformations: Specifically, Eqs. (3.6), which is simply Eqs. (2.1) and (2.2) with the weight $\bar{\sigma}^{-1}$ entirely ignored, produces parts which transform as Eqs. (2.6) and (2.11) under conformal transformations. The two parts, therefore, do not transform alike.

The fact that Eqs. (2.6) and (2.11) behave differently under conformal transformations leads to two inequivalent methods of decomposing \bar{A}^{ij} . What Isenberg [9] has called “Method A” is first to transform \bar{A}^{ij} conformally, then split it with σ and $\bar{\sigma}$ ignored, then transform the unbarred vector part with the “wrong” transformation (using φ^{-10} instead of φ^{-4} so as to scale its divergence). This is discussed in [1]. “Method B” is, in effect, to split \bar{A}^{ij} first, then transform conformally to obtain a slightly different, and more difficult, form of the constraints [2]. The existence of *two* methods of dealing with the extrinsic curvature using the old tensor splitting suggests that neither is the optimal method. The method introduced here has no such ambiguity and can be regarded as the resolution (in the conformal framework) of the initial value problem in the extrinsic curvature representation.

We apply the splitting (2.2) to the vector constraint (3.4). After the conformal transformations, we obtain

$$\nabla_j[\sigma^{-1}(\text{LB})^{ij}] = \frac{2}{3}\varphi^6\nabla^i\tau, \quad (3.7)$$

where we used

$$A^{ij} = A_{TT}^{ij} + \sigma^{-1}(\text{LB})^{ij}. \quad (3.8)$$

Equation (3.7) is the momentum constraint. We will see that it will determine B^i once σ is chosen. The term A_{TT}^{ij} disappears from the vector constraint. It is “freely specifiable” and can be determined by extracting the TT part of some traceless symmetric tensor C^{ij} :

$$A_{TT}^{ij} = C^{ij} - \sigma^{-1}(\text{LV})^{ij}, \quad (3.9)$$

where C^{ij} is freely given, as is $\sigma > 0$, and V^i is then determined by solving an equation similar to Eq. (2.4). In this step boundary conditions must be applied when \mathcal{M} has boundaries, which will influence V^i and A_{TT}^{ij} .

Thus C^{ij} supplies a “source” in the vector constraint (3.4), which, if we define [10,11]

$$X^i = B^i - V^i, \quad (3.10)$$

becomes (in vacuum)

$$\nabla_j[\sigma^{-1}(\text{LX})^{ij}] = \frac{2}{3}\varphi^6\nabla^i\tau - \nabla_j C^{ij}. \quad (3.11)$$

From its solution we can construct

$$A^{ij} = C^{ij} + \sigma^{-1}(\text{LX})^{ij} \quad (3.12)$$

$$= A_{TT}^{ij} + \sigma^{-1}(\text{LB})^{ij}. \quad (3.13)$$

Next, recall that $\bar{g}_{ij} = \varphi^4 g_{ij}$ implies [12]

$$\bar{R} = R\varphi^{-4} - 8\varphi^{-5}\Delta\varphi, \quad (3.14)$$

where R is the scalar curvature of g_{ij} . Equation (3.14) enables us to rewrite the scalar constraint (3.5) in transformed variables as

$$\Delta\varphi - \frac{1}{8}R\varphi = -\frac{1}{8}A_{ij}A^{ij}\varphi^{-7} + \frac{1}{12}\tau^2\varphi^5. \quad (3.15)$$

[We also used Eq. (2.13), which implies $\bar{A}_{ij} = \varphi^{-2}A_{ij}$.]

IV. THE WEIGHT FUNCTION AND THE LAPSE FUNCTION

The extrinsic curvature formulation of the initial value problem uses, in essence, the canonical variables. Furthermore, it depends only on the embedding (encoded by \bar{K}_{ij}) of the hypersurface (\mathcal{M}, \bar{g}) into the four-dimensional spacetime $(\mathcal{V}, \mathbf{g})$. It does not depend on the foliation or the time vector $\partial/\partial t$, that is, lapse \bar{N} and shift β^i . (This is well known but can still be a source of confusion. Therefore, we review the second fundamental form and the extrinsic curvature in the Appendix.) Such a statement cannot be made in the thin sandwich formulation, where one is interested precisely in the extension along curves tangent to $\partial/\partial t$.

Since the establishment of the canonical form of the action by Arnowitt, Deser, and Misner (ADM) [13], the shift β^i has been taken to be the undetermined multiplier of the momentum (vector) constraint, while the lapse has been taken to be the undetermined multiplier of the Hamiltonian (scalar) constraint. However, it has come to light that the lapse as a multiplier must be replaced by the *lapse antidensity* α , a scalar of weight (-1) [14–17]. This replacement is required in order that the canonical framework as a whole for Einsteinian gravity makes complete sense, that is, that it works in the same way as for other physical systems that can be derived from an action principle. For technical details, see [17,18]. We are requiring essentially just that the “Hamiltonian vector field” be defined without reference to the constraints in the whole phase space of g ’s and π ’s.

That α and β^i are undetermined multipliers of the vector and scalar constraints means that they are both conformally invariant: $\bar{\alpha} = \alpha$ and $\bar{\beta}^i = \beta^i$. But the invariance of α has very interesting consequences. *When the scalar constraint is satisfied*, then upon examination of the ADM action, we see that $\alpha = \bar{N}\bar{g}^{-1/2}$. Hence, $\bar{\alpha} = \alpha$ implies that

$$\bar{N} = \varphi^6 N \quad (4.1)$$

because $\bar{g}^{1/2} = \varphi^6 g^{1/2}$. Thus, the physical lapse is not quite arbitrary for our purpose of solving the constraints. We have a “trial” lapse N and a final physical lapse \bar{N} that will be determined by N and the solution φ of the scalar constraint.

Recall, that in studies of hyperbolic forms of the Einstein evolution equations with *physical* characteristic speeds, it is also found that α , not \bar{N} is arbitrary (see, e.g., [19,20]).

From Eq. (4.1) we observe that \bar{N} and N are related just as were $\bar{\sigma}$ and σ . Thus we have a natural geometrical choice:

$$\bar{\sigma} = 2\bar{N}, \quad \sigma = 2N. \quad (4.2)$$

The factors “2” are chosen for later convenience. As consequence of Eqs. (4.2), \bar{N} appears in \bar{A}^{ij} and \bar{K}^{ij} . Before, however, we have noted that \bar{K}^{ij} is independent of \bar{N} . But our use of \bar{N} only determines the *splitting* of \bar{K}^{ij} , not \bar{K}^{ij} itself.

Let us examine the consequences of the choices (4.2). Constraints (3.11) and (3.15) become

$$\nabla_j[(2N)^{-1}(\mathbb{L}X)^{ij}] = \frac{2}{3}\varphi^6\nabla^i\tau - \nabla_j C^{ij}, \quad (4.3)$$

$$\Delta\varphi - \frac{1}{8}R\varphi = \frac{1}{12}\tau^2\varphi^5 - \frac{1}{8}A_{ij}A^{ij}\varphi^{-7}, \quad (4.4)$$

where A_{ij} is given by Eq. (3.12) with $(2N)$ replacing σ .

Of course, N itself is yet to be chosen. But we have several remarkable automatic consequences of solving Eqs. (4.3) and (4.4) for X^i and φ , given a uniformly positive N and supposing φ is also uniformly positive.

(1) We have from Eq. (4.2) and the conformal transformation rules, that

$$A^{ij} = A_{TT}^{ij} + (2N)^{-1}(\mathbb{L}B)^{ij} \quad (4.5)$$

implies

$$\bar{A}^{ij} = \bar{A}_{TT}^{ij} + (2\bar{N})^{-1}(\bar{\mathbb{L}}B)^{ij}. \quad (4.6)$$

The consequence is that \bar{A}_{TT}^{ij} and $(\bar{\mathbb{L}}B)^{ij}$ are now orthogonal in the measure $\mu_{\bar{g},\bar{N}} = \bar{N}\bar{g}^{1/2}d^3x = \sqrt{-\mathbf{g}}d^3x$, where \mathbf{g} is the determinant of the physical *spacetime* metric. This is the spacetime measure (apart from “ dt ”), and so *orthogonality* is determined by the spacetime geometry of \mathcal{M} , including the lapse function, embedded in \mathcal{V} . This result is independent of the method of determination of N .

(2) Whatever the result for φ , X^i , and $\bar{N} = \varphi^6 N$, we can show consistency of the results with the conformal thin sandwich equations [3]. Consider the traceless metric velocity

$$\bar{u}_{ij} = \partial_t \bar{g}_{ij} - \frac{1}{3}\bar{g}_{ij}\bar{g}^{kl}\partial_t \bar{g}_{kl}. \quad (4.7)$$

Using the evolution equation (A18), as well as Eq. (4.6), we find

$$\bar{u}^{ij} = -2\bar{N}[\bar{A}_{TT}^{ij} + (2\bar{N})^{-1}(\bar{\mathbb{L}}B)^{ij}] + (\bar{\mathbb{L}}\beta)^{ij}, \quad (4.8)$$

where β^i is the shift, which, like the lapse antidensity $\alpha = Ng^{-1/2}$ can be freely chosen. The shift *vector* is conformally invariant. Therefore,

$$\bar{u}^{ij} = -(2N\varphi^6)(\varphi^{-10}A_{TT}^{ij}) - \varphi^{-4}[\mathbb{L}(B - \beta)]^{ij} \quad (4.9)$$

$$= -\varphi^{-4}(2NA_{TT}^{ij} + [\mathbb{L}(B - \beta)]^{ij}) \quad (4.10)$$

$$= -\varphi^{-4}(2NC^{ij} + [\mathbb{L}(B - V - \beta)]^{ij}). \quad (4.11)$$

We see that three vectors enter \bar{u}^{ij} : β^i , a gauge choice, V^i , which removes the longitudinal piece of C^{ij} , and B^i , which solves the vector constraint. Let us set $B^j - V^j - \beta^j = Z^j$. By the choice of shift $\beta^j = B^j - V^j$, we could render $Z^j = 0$. Then $\bar{u}^{ij} = -\varphi^{-4}(2NC^{ij})$ and $\bar{u}_{ij} = -\varphi^4(2NC_{ij})$. Furthermore, with the choice $\beta^i = B^i - V^i = X^i$, the constraints (3.11) and (3.15) are *identical* to the constraint equations in the conformal thin sandwich formalism [Eqs. (14) and (15) of [3]], provided one identifies $-2NC_{ij}$ with the conformal metric velocity u_{ij} .

(3) With $\alpha(x, t)$ given, and \bar{N} determined by $\alpha \equiv \bar{g}^{-1/2}\bar{N}$, we find that \bar{N} always obeys a generalized harmonic evolution,

$$\hat{\partial}_0 \bar{N} + \bar{N}^2 \tau = \bar{N} \hat{\partial}_0 \log \alpha, \quad (4.12)$$

where $\hat{\partial}_0 = \partial_t - \mathcal{L}_\beta$.

The identification $\sigma = 2N$ still leaves us with the question of how to choose a trial lapse function in a reasonable way. In principle, any choice will do and we will know the geometric meaning of $\bar{N} = \varphi^6 N$.

The choice $N = 1$, for example, is allowed in the extrinsic curvature representation. Indeed, $N = 1$ would then give back method A, but now that the behavior of the lapse function is understood, it gives a correct $\partial_t g_{ij}$. It also tells us $\bar{N} = \varphi^6 = (\bar{g}/g)^{1/2}$. Another choice of N is discussed in Sec. VI below.

In picking the “true degrees of freedom,” that is, the conformal class of the metric and \bar{K}_{ij}^{TT} , and the mean curvature τ , the lapse intervenes in pinpointing \bar{K}_{ij}^{TT} . There are infinitely many choices other than $N = 1$. This feature did not arise in previous studies of the extrinsic curvature picture; but, on the other hand, those pictures do not in general fit the purely geometric construction of the conformal thin sandwich equations. In curved spacetime, the dynamical degrees of freedom in the Hamiltonian picture do not allow themselves to be identified without the foliation being inextricably involved. Another consequence is that time is not found among the traditional canonical variables alone, even in general relativity.

V. STATIONARY SPACETIMES

Consider a stationary solution of Einstein’s equations with timelike Killing vector t . Given a spacelike hypersurface Σ , there is a preferred gauge such that the time vector of an evolution on Σ coincides with t , namely $\bar{N} = -\langle n, t \rangle_{\mathbf{g}}$, $\beta = \perp t$, where n is the unit normal to Σ , and \perp is the

projection operator into Σ , $\langle \perp, n \rangle_g = 0$. With this choice of lapse and shift, \bar{g}_{ij} and \bar{K}_{ij} will be independent of time. Using $\partial_t \bar{g}_{ij} = 0$ in Eq. (A17) yields

$$\bar{K}_{ij} = \frac{1}{2\bar{N}} (\nabla_i \bar{\beta}_j + \nabla_j \bar{\beta}_i), \quad (5.1)$$

with $\bar{\beta}_i = \bar{g}_{ij} \beta^j$. The tracefree part of this equation implies

$$\bar{A}^{ij} = \frac{1}{2\bar{N}} (\bar{L}\beta)^{ij}. \quad (5.2)$$

Therefore, *with* the appropriate weight factor $\bar{\sigma} = 2\bar{N}$ as constructed above, the extrinsic curvature (2.2) has *no transverse traceless piece* for any spacelike slice in any spacetime with a timelike Killing vector.² A transverse-traceless decomposition of \bar{A}^{ij} *without* the weight factor, however, will in general lead to a nonzero transverse traceless piece. This was previously a puzzle. The TT part is generally identified with the dynamical degrees of freedom. Therefore the radiative aspect of a stationary (or static) spacetime should be manifestly zero on a natural slicing associated with the timelike Killing vector. The absence of this property for stationary, nonstatic, spacetimes with previous decompositions was a serious weakness.

These considerations provide an independent argument for the introduction of the weight function in Eq. (2.2), and the identification of $\bar{\sigma}$ with the lapse function in Eq. (4.2).

VI. A GEOMETRICAL CHOICE OF N AND \bar{N} : THE CONFORMAL THIN SANDWICH VIEWPOINT

The *conformal thin sandwich equations* [3] specify freely (1) a conformal metric g_{ij} and its velocity $\partial_t g_{ij} = \dot{g}_{ij}$, and (2) the mean curvature τ . The lapse \bar{N} is $\varphi^6 N$ with N still adrift. But there is a definitive solution for fixing N : The mean curvature has become a configuration variable, for which a value and a velocity need to be specified; as one must specify g_{ij} and its velocity \dot{g}_{ij} , by analogy one can give the mean curvature τ and its velocity $\partial_t \tau = \dot{\tau}$. This will determine both N and \bar{N} .

The specification

$$(g_{ij}, \dot{g}_{ij}; \tau, \dot{\tau}) \quad (6.1)$$

has the same *number* of variables as the conventional choice

$$(\bar{g}_{ij}, \dot{\bar{g}}_{ij}) = (g_{ij}, \dot{g}_{ij}; \bar{g}^{1/2}, \dot{\bar{g}}^{1/2}). \quad (6.2)$$

Furthermore $\bar{g}^{1/2}$ and τ are *canonically conjugate* (apart from an irrelevant constant), so that Eqs. (6.1) and (6.2) are as

²A similar argument is applicable in the ergosphere of a Kerr black hole; however, one must be more careful with the choice of Σ relative to t .

close and as symmetric to each other as possible. However, the conventional specification (6.2) fails [21] while the conformal one (6.1) does not fail.

Thus, in the conformal thin sandwich problem, we also give

$$f[\beta; t, x] = \hat{\partial}_0 \tau = \partial_t \tau - \beta^i \partial_i \tau \quad (6.3)$$

$$= -\bar{\Delta} \bar{N} + (\bar{R} + \tau^2) \bar{N} \quad (6.4)$$

which is an Einstein equation on \mathcal{M} , conventionally regarded as an equation for $\partial_t^2 \bar{g}^{1/2}$. (Note that $\partial_t \bar{g}^{1/2} \sim \tau$ is not a constraint; it is an identity, part of the definition of extrinsic curvature [cf. Eq. (A18)].) We are turning the equation of motion for $\bar{g}^{1/2}$ into a constraint. This is an old “trick;” $\tau = \dot{\tau} = 0$ is maximal slicing [12], whereas τ and $\dot{\tau}$ constant in space (but allowing changes in time) is constant mean curvature slicing [22]. Also, $\dot{\tau} = 0$ is used during construction of quasiequilibrium initial data (see e.g. [23–25] and references in [11]). However, now it is clear that specification of $\dot{\tau}$ is fundamentally linked to the initial value problem.

We saw in the previous section that the choice of slicing (\bar{N} , the passing of time) enters into the *construction* of the $(\bar{g}_{ij}, \bar{K}_{ij})$ representation. In addition, this “Hamiltonian” representation is consistent with the “Lagrangian” conformal thin sandwich picture. Hence we can adopt Eq. (6.4) for the $(\bar{g}_{ij}, \bar{K}_{ij})$ representation as well. In other representations of the extrinsic curvature in this problem (methods A and B), there is no \bar{N} ; and the construction has nothing to say about the passage of time. One does not move forward in time without an extra equation to give \bar{N} . Recall $d\tau^{(\text{prop})}/dt = \bar{N}$, where $\tau^{(\text{prop})}$ is a local Cauchy observer’s proper time and t is the coordinate time.

Using the scalar constraint (3.3) in Eq. (6.4) yields [see e.g. Eq. (98) of [1]]

$$f[\beta; t, x] = -\bar{\Delta} \bar{N} + \bar{K}_{ij} \bar{K}^{ij} \bar{N} = -\bar{\Delta} \bar{N} + \left(\bar{A}_{ij} \bar{A}^{ij} + \frac{1}{3} \tau^2 \right) \bar{N}. \quad (6.5)$$

Conformal transformation of the Laplacian is carried out by expressing it as

$$\bar{\Delta}(\cdot) = \bar{g}^{-1/2} \partial_i [\bar{g}^{1/2} \bar{g}^{ij} \partial_j (\cdot)]. \quad (6.6)$$

Preliminarily, we find

$$\begin{aligned} f[\beta; t, x] &= -\varphi^{-4} \Delta(N\varphi^6) - 2\varphi^{-5} (\nabla_i \varphi) [\nabla^i (N\varphi^6)] \\ &\quad + \left(A_{ij} A^{ij} \varphi^{-12} + \frac{1}{3} \tau^2 \right) (N\varphi^6). \end{aligned} \quad (6.7)$$

To proceed, we use the scalar constraint in conformal form, Eq. (3.15), to eliminate $\Delta\varphi$ and find

$$\Delta N - \left[\frac{7}{4} A_{ij} A^{ij} \varphi^{-8} - \frac{1}{6} \tau^2 \varphi^4 - \frac{3}{4} R - 42 (\nabla_i \log \varphi)^2 \right] N + 14 (\nabla_i N) (\nabla^i \log \varphi) + \varphi^{-2} (\partial_i \tau - \beta^i \partial_i \tau) = 0, \quad (6.8)$$

where $\tau(t, x)$ is given.

Equation (6.8) is a fifth elliptic equation, coupled to the others, that is required for the completeness of the conformal thin sandwich equations and is also natural in the extrinsic curvature representation given here.

[Though the four conventional thin sandwich equations do not work, it is interesting that in the Baierlein-Sharp-Wheeler (BSW) treatment [26], there is an implicit fifth equation. Differentiation of the second order equation for the shift, with a given lapse, produces a first order, *not* a third order, equation: an integrability condition. This was discovered by Pereira [27].]

VII. CONCLUSION

By a simple tensor decomposition, one can bring the constraint equations in the extrinsic curvature form into geometrical and mathematical conformity with the conformal thin sandwich equations when N is arbitrary in both sets of equations. This statement remains true if N is fixed by the same method in both formulations.

The conformal thin sandwich equations with $\dot{\tau}$ fixed form an elliptic system whose general properties remain to be studied. We expect that it is a solvable, solid, system provided that τ and $\dot{\tau}$ do not vary wildly.

A natural choice of \bar{N} in stationary spacetimes has been seen to render the transverse traceless part of \bar{K}^{ij} zero. The most striking result of the present analysis is the inextricable relation of time, represented by \bar{N} or $\dot{\tau}$, and space, represented by the four constraint equations.

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APPENDIX: SECOND FUNDAMENTAL FORM AND EXTRINSIC CURVATURE

Let \mathcal{M} be an m -dimensional surface embedded in a d -dimensional ambient space \mathcal{V} (we do not assume that $d = m + 1$). Let \mathcal{V} be endowed with a Riemannian or Lorentzian metric \mathbf{g} and corresponding Levi-Civita connection D , while \mathcal{M} inherits a Riemannian metric g and connection ∇ .

Let X and Y be vectors in \mathcal{V} that are tangent to \mathcal{M} . The first fundamental form of \mathcal{M} for X and Y is $g(X, Y)$, while the second fundamental form of \mathcal{M} with respect to X and Y is the vector [28]

$$h(X, Y) \equiv \nabla_X Y - D_X Y. \quad (A1)$$

The *purpose of the second fundamental form* is to discriminate between parallel transport of a vector Y along the direction of a vector X in the $(\mathcal{V}, \mathbf{g}, D)$ connection and in the (\mathcal{M}, g, ∇) connection, when both X and Y are tangent to \mathcal{M} . This is defined without reference to surfaces near \mathcal{M} or a foliation. It tells us from the viewpoint of \mathcal{V} , whether, say, the geodesics of \mathcal{M} also appear “straight” in \mathcal{V} .

Zero torsion in the Levi-Civita connections D and ∇ implies

$$h(X, Y) = h(Y, X). \quad (A2)$$

Now we demonstrate that $h(X, Y)$ is always orthogonal to \mathcal{M} . Suppose X , Y , and Z are tangent to \mathcal{M} and consider \mathcal{V} 's scalar product between vectors $\langle \cdot, \cdot \rangle = \mathbf{g}(\cdot, \cdot)$. The product rule for derivatives using D_X gives

$$\langle D_X Y, Z \rangle = X \langle Y, Z \rangle - \langle Y, D_X Z \rangle. \quad (A3)$$

A similar rearrangement using ∇_X gives the same expression with D_X replaced by ∇_X . Combining the two expressions so as to cancel the common term $X \langle Y, Z \rangle$, and invoking Eqs. (A1) and (A2) yields

$$\langle h(X, Y), Z \rangle = -\langle h(Z, X), Y \rangle. \quad (A4)$$

Hence, the trilinear form on the left changes sign under a cyclic permutation of X , Y , and Z . But three such permutations restore the original order, which must then be the negative of itself. Hence, it is zero, and thus $h(X, Y)$ is orthogonal to \mathcal{M} .

Let us state the consequences in tensor language by supposing that \mathcal{M} has an adapted basis e_i ($i, j, \dots = 1, 2, \dots, m$). The full basis of \mathcal{V} is e_α ($\alpha, \beta, \dots = 1, 2, \dots, d$), where the first m vectors are the e_i .

We define the coefficients of the connection one-forms of $(\mathcal{V}, \mathbf{g}, D)$ by

$$D_{e_\alpha} e_\beta = \omega_{\alpha\beta}^\gamma e_\gamma \quad (A5)$$

[this differs from the convention of Misner, Thorne, and Wheeler (MTW) [29]]. After a brief calculation we find that $h(X, Y)$ can be written as

$$h(X, Y) = - \sum_{\alpha=m+1}^d e_\alpha \omega_{ij}^\alpha X^i Y^j. \quad (A6)$$

We expand the basis of the co-space of \mathcal{M} in \mathcal{V} in terms of $(d-m)$ mutually orthogonal unit normals to \mathcal{M} , $n_{\hat{a}}$, $\hat{a} = \hat{m} + \hat{1}, \dots, \hat{d}$. There is a $(d-m) \times (d-m)$ nonsingular matrix $E_{\hat{a}}^\alpha$ such that

$$e_\alpha = \sum_{\hat{a}=\hat{m}+\hat{1}}^{\hat{d}} E_{\hat{a}}^\alpha n_{\hat{a}}, \quad \alpha = m+1, \dots, d. \quad (A7)$$

Hence, Eq. (A6) becomes

$$h(X, Y) = - \sum_{\alpha=m+1}^d \sum_{\hat{a}=m+1}^{\hat{d}} n_{\hat{a}} (E_{\alpha}^{\hat{a}} \omega_{ij}^{\alpha}) X^i Y^j. \quad (\text{A8})$$

The $d-m$ extrinsic curvature tensors $K_{ij}^{\hat{a}}$ are defined by

$$K_{ij}^{\hat{a}} = - \sum_{\alpha=m+1}^d E_{\alpha}^{\hat{a}} \omega_{ij}^{\alpha}. \quad (\text{A9})$$

Equations (A9) and (A1) emphasize that the extrinsic curvature is related to transport *parallel* to the slice, a viewpoint not present in a definition in terms of derivatives *normal* to the slice. But both definitions agree.

Now let \mathbf{g} be Lorentzian, g Riemannian, and $m=d-1$. We are interested in the case $d=4$, $m=3$. In this case \mathcal{M} is a hypersurface, which we take to be $t=\text{const}$. There is only one extrinsic curvature tensor K_{ij} . The spacetime metric is

$$\mathbf{g} = -N^2 dt^2 + g_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \quad (\text{A10})$$

where N is the lapse and β^i the shift. We choose the “Cauchy-adapted” coframe θ^{α} ($\alpha, \beta, \dots = 0, 1, 2, 3$; $i, j, \dots = 1, 2, 3$):

$$\theta^0 = dt, \quad \theta^i = dx^i + \beta^i dt. \quad (\text{A11})$$

The dual vector frame is $e_{\alpha} = \partial_{\alpha}$, with

$$\partial_0 = \partial_t - \beta^i \partial_i, \quad \partial_i = \frac{\partial}{\partial x^i}. \quad (\text{A12})$$

We use ∂ to denote Pfaffian derivatives, some of which are natural (namely $\partial_i = \partial/\partial x^i$ and $\partial_t = \partial/\partial t$). In particular, the spatial basis ∂_i is natural, so that the connection coefficients and Christoffel symbols of g_{ij} are equal.

For the hypersurface $t=\text{const}$ we find

$$\nabla_X Y - D_X Y = -X^i Y^j (\omega_{ij}^0 e_0) = -X^i Y^j (N \omega_{ij}^0) n_{\hat{0}}, \quad (\text{A13})$$

where we used $e_0 = N n_{\hat{0}}$. ω_{ij}^0 can be evaluated by the formula

$$\omega_{\beta\gamma}^{\alpha} = \Gamma_{\beta\gamma}^{\alpha} + \frac{1}{2} \mathbf{g}^{\alpha\delta} (C_{\delta\beta}^{\lambda} \mathbf{g}_{\gamma\lambda} + C_{\delta\gamma}^{\lambda} \mathbf{g}_{\beta\lambda}) + \frac{1}{2} C_{\beta\gamma}^{\alpha}, \quad (\text{A14})$$

where the structure coefficients $C_{\beta\gamma}^{\alpha}$ are defined by

$$d\theta^{\alpha} = -\frac{1}{2} C_{\beta\gamma}^{\alpha} \theta^{\beta} \wedge \theta^{\gamma}, \quad [e_{\alpha}, e_{\beta}] = C_{\alpha\beta}^{\gamma} e_{\gamma}. \quad (\text{A15})$$

In our frame, all $C_{\beta\gamma}^{\alpha}$ vanish except $C_{0j}^i = -C_{j0}^i = \partial_j \beta^i$, and one finds

$$K_{ij} \equiv K_{ij}^{\hat{0}} = -N \omega_{ij}^0 = -\frac{1}{2} N^{-1} (\partial_t - \mathcal{L}_{\beta}) g_{ij}. \quad (\text{A16})$$

Here, \mathcal{L}_{β} is the spatial Lie derivative along the shift vector. Since $\mathcal{L}_{\beta} g_{ij} = \nabla_i \beta_j + \nabla_j \beta_i$ with $\beta_i \equiv g_{ij} \beta^j$, Eq. (A16) gives

$$K_{ij} = -\frac{1}{2} N^{-1} (\partial_t g_{ij} - \nabla_i \beta_j - \nabla_j \beta_i). \quad (\text{A17})$$

Rewriting Eq. (A17) gives

$$\partial_t g_{ij} = -2N K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i. \quad (\text{A18})$$

If you prefer the opposite sign for K_{ij} , as some authors do, simply change the sign of $h(X, Y)$ in its definition. Equations (A17) and (A18) change sign when passing from the Lorentzian to the Riemannian (“Euclidean”) case for either choice of the sign of $h(X, Y)$.

For completeness, we give all connection coefficients of the frame defined by Eqs. (A10), (A11) and (A12):

$$\omega_{00}^0 = \partial_0 \log N, \quad \omega_{ij}^0 = -N^{-1} K_{ij}, \quad (\text{A19})$$

$$\omega_{j0}^i = -N K_j^i, \quad \omega_{0j}^i = -N K_j^i + \partial_j \beta^i, \quad (\text{A20})$$

$$\omega_{i0}^0 = \omega_{0i}^0 = \partial_i \log N, \quad \omega_{00}^i = N g^{ij} \partial_j N, \quad (\text{A21})$$

$$\omega_{jk}^i = \Gamma_{jk}^i. \quad (\text{A22})$$

We have rewritten time derivatives in terms of the extrinsic curvature; Γ_{jk}^i denotes the Christoffel symbols of the spatial metric g_{ij} . We note again that we do not use the MTW convention for the order of indices of the connection coefficients. In our frame, this is significant only for Eq. (A20).

- [1] J. W. York, Jr., in *Sources of Gravitational Radiation*, edited by L. L. Smarr (Cambridge University Press, Cambridge, England, 1979), p. 83.
- [2] N.Ó. Murchadha and J.W. York, Jr., Phys. Rev. D **10**, 428 (1974).
- [3] J.W. York, Jr., Phys. Rev. Lett. **82**, 1350 (1999).
- [4] S. Deser, Ann. Inst. Henri Poincaré, Sect. A **7**, 149 (1967).
- [5] J.W. York, Jr., J. Math. Phys. **14**, 456 (1973).
- [6] J.W. York, Jr., Ann. Inst. Henri Poincaré, Sect. A **21**, 319 (1974).

- [7] H.P. Pfeiffer, G.B. Cook, and S.A. Teukolsky, Phys. Rev. D **66**, 024047 (2002).
- [8] N. Ó. Murchadha (private communication).
- [9] J. Isenberg (private communication).
- [10] M. Cantor (private communication).
- [11] G.B. Cook, Living Rev. Relativ. **3** (2000); URL <http://livingreviews.org/Articles/Volume3/2000-5cook/>
- [12] A. Lichnerowicz, J. Math. Pures Appl. **23**, 37 (1944).
- [13] R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley,

- New York, 1962), p. 227.
- [14] Y. Choquet-Bruhat and T. Ruggeri, *Commun. Math. Phys.* **89**, 269 (1983).
 - [15] C. Teitelboim, *Phys. Rev. D* **28**, 297 (1983).
 - [16] A. Ashtekar, *Phys. Rev. D* **36**, 1587 (1987).
 - [17] A. Anderson and J.W. York, Jr., *Phys. Rev. Lett.* **81**, 1154 (1998).
 - [18] J. W. York, Jr., in *Relativity, Particle Physics and Cosmology*, edited by R. E. Allen (World Scientific, Singapore, 1999), pp. 77–85.
 - [19] Y. Choquet-Bruhat and J.W. York, *Banach Center Publ.* **41**, 119 (1997).
 - [20] L.E. Kidder, M.A. Scheel, and S.A. Teukolsky, *Phys. Rev. D* **64**, 064017 (2001).
 - [21] R. Bartnik and G. Fodor, *Phys. Rev. D* **48**, 3596 (1993).
 - [22] J.W. York, Jr., *Phys. Rev. Lett.* **28**, 1082 (1972).
 - [23] J. R. Wilson and G. J. Mathews, in *Frontiers in Numerical Relativity*, edited by C. R. Evans, L. S. Finn, and D. W. Hobill (Cambridge University Press, Cambridge, England, 1989), pp. 306–314.
 - [24] E.ourgoulhon, P. Grandclément, and S. Bonazzola, *Phys. Rev. D* **65**, 044020 (2002).
 - [25] G.B. Cook, *Phys. Rev. D* **65**, 084003 (2002).
 - [26] R.F. Baierlein, D.H. Sharp, and J.A. Wheeler, *Phys. Rev.* **126**, 1864 (1962).
 - [27] C. Pereira, *J. Math. Phys.* **14**, 1498 (1973).
 - [28] M. E. Taylor, *Partial Differential Equations II* (Springer, Berlin, 1996), p. 479.
 - [29] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, New York, 1973).